Διαφορικές εξισώσεις 2ο μάθημα No. Date 10-10-2019

$y' = f(x,y)$, $x \in I$ διάστημα του $\mathbb{R}$

*** Άμεση ολοκλήρωση**

$$y'(x) = \frac{p(x)}{g(y(x))}, x \in I \quad g(y(x)) \neq 0$$

τότε $\quad p(x) = g(y(x))\, y'(x), \ x \in I$

Παίρνω $x_0 \in I$, $x \in I$

$$\int_{x_0}^{x} g(y(s))\, y'(s)\, ds = \int_{x_0}^{x} p(s)\, ds \quad (1)$$

Υποθέτουμε ότι όλα είναι συνεχείς συναρτήσεις

θέτουμε $\quad y(s) = u \quad y'(s)ds = du \Rightarrow ds = \frac{du}{y'(s)} \quad (2)$

$$(1) \overset{(2)}{\Rightarrow} \int_{y(x_0)}^{y(x)} g(u)\, du = \int_{x_0}^{x} p(s)\, ds \quad (3) \qquad \text{θέτουμε } G'(u) = g(u)$$

Τότε $\quad (3) \Rightarrow G(y(x)) - G(y(x_0)) = \int_{x_0}^{x} p(s)\, ds$

Άρα $\quad G(y(x)) = \underbrace{G(y(x_0))}_{c} + \int_{x_0}^{x} p(s)\, ds, \ x \in I$

Παράδειγμα

$$y'(x) = \frac{x^3 + \cos x}{y^2(x) + c}, \ x_0 \in [0, 2019] = I.$$

$y^2(x) + c \neq 0 \quad \forall x \in I$

$y'(x)(y^2(x) + c) = x^3 + \cos x$

No. Date

$$\int_{x_0}^{x} y'(s)\,(y^2(s)+c)\,ds = \int_{x_0}^{x} (s^3+\cos s)\,ds$$

Θέτουμε $y(s) = u$, $y'(s)ds = du$

- $s = x \Rightarrow u = y(x)$
- $s = x_0 \Rightarrow u = y(x_0)$

$$\int_{g(x_0)}^{g(x)} (u^2+c)\,du = \left[\frac{s^4}{4} + \sin s\right]_{x_0}^{x} \Rightarrow \left[\frac{u^3}{3} + uc\right]_{y(x_0)}^{y(x)}$$

$$= \frac{x^4}{4} + \sin x - \frac{x_0^4}{4} - \sin x_0 = \frac{y^3(x)}{3} + y(x)\cdot c - \frac{y(x_0)^3}{3} - y(x_0)\cdot c$$

$$= \frac{x^4}{4} + \sin x - \frac{x_0^4}{4} - \sin x$$

$$= \frac{y^3(x)}{3} - \frac{y(0)^3}{3} = \frac{x^4}{4} + \sin x = g^3(x) = y(0)^3 + \frac{3x^4}{4} + 3\sin x$$

Άρα $y(x) = \sqrt[3]{y(0)^3 + \frac{3x^4}{4} + 3\sin x}$, $x \in I$

αφού $c = 0 \sim y^2(x) \neq 0 \Rightarrow \boxed{y(x) \neq 0}$

Γραμμική εξίσωση Πρώτης τάξης
$a_1(x)\,y'(x) + a_0(x)\,y(x) = b(x)$, $x_0, x \in I$ $a_1, a_0, b \in C$

Αν $b = b(x) = 0 \sim$ εξίσωση ομογενής

Αν $b = b(x) \neq 0 \sim$ μη ομογενής εξίσωση

Θέλουμε ο συντελεστής του $y'(x)$ να είναι διάφορος του μηδενός

http://www.tanenghong.com

No. Date

$\boxed{\text{Av } b=0} \rightsquigarrow \alpha_1(x)y'(x)+\alpha_0(x)y(x)=0$

$$\frac{y'(x)}{y(x)} = \frac{-a_0(x)}{\alpha_1(x)} = \int \frac{y'(x)}{y(x)}\,dx = \int \frac{-a_0(x)}{\alpha_1(x)}\,dx$$

$$= \ln|y(x)| = -\int \frac{\alpha_0(x)}{\alpha_1(x)}\,dx + C$$

$$|y(x)| = e^{C-\int \frac{\alpha_0(x)}{\alpha_1(x)}dx} \Rightarrow y(x) = \pm e^{C-\int \frac{\alpha_0(x)}{\alpha_1(x)}dx}$$

$\boxed{\text{Av } b\neq 0}$

$$y'(x) + \frac{\alpha_0(x)}{\alpha_1(x)}y(x) = \frac{b(x)}{\alpha_1(x)} =$$

$$e^{\int\frac{\alpha_0(x)}{\alpha_1(x)}} y'(x) + e^{\int\frac{\alpha_0(x)}{\alpha_1(x)}dx} \frac{\alpha_0(x)}{\alpha_1(x)} y(x) = e^{\int \frac{\alpha_0(x)}{\alpha_1(x)}dx} \cdot \frac{b(x)}{\alpha_1(x)}.$$

ooo $y(x) = \ldots\ldots, x \in I.$

$$y(x) = e^{-\int_{x_0}^{x}\frac{\alpha_0(s)}{\alpha_1(s)}ds}\left(y(x_0) + \int_{x_0}^{x} \frac{b(u)}{\alpha_1(u)} \cdot e^{\int_{x_0}^{u}\frac{\alpha_0(s)}{\alpha_1(s)}ds} du\right)$$

$y'(x)+p(x)y(x)=q(x),\ p,q \in C(I),\ x_0, x \in I,$

$$y(x) = e^{-\int_{x_0}^{x}p(s)ds}\left(y(x_0) + \int_{x_0}^{x} q(s)\cdot e^{\int_{x_0}^{s}p(u)du} ds\right), x \in I$$

No. Date

Παράδειγμα:

$xy' - 2y = -x^2, \quad y(1)=0$

$xy'(x) - 2y(x) = -x^2, \quad x \neq 0, x>0.$

$y'(x) - \frac{2}{x} y(x) = -x$,

$e^{-2\ln|x|} y'(x) - e^{-2\ln|x|} \cdot \frac{2}{x} y(x) = -x \cdot e^{-2\ln|x|}$

$\Rightarrow e^{\ln\left(\frac{1}{x^2}\right)} y'(x) - e^{\ln\frac{1}{x^2}} \cdot \frac{2}{x} y(x) = -x e^{\ln\frac{1}{x^2}}$

$\Rightarrow \frac{1}{x^2} y'(x) - \frac{2}{x^3} y(x) = -x \cdot \frac{1}{x^2} \Rightarrow$

$\Rightarrow \frac{1}{x^2} y'(x) - \frac{2}{x^3} y(x) = -\frac{1}{x}$

$\left(\frac{1}{x^2} y(x)\right)' = -\frac{1}{x}$

$\frac{1}{x^2} y(x) = -\ln|x| + C \overset{x>0}{\Rightarrow} y(x) = -x^2 \ln x + x^2 C$

$x=1 \rightsquigarrow y(1) = -1 \cdot \ln 1 + 1^2 \cdot C \Rightarrow \boxed{0 = C}$

Άρα $y(x) = -x^2 \ln x, \quad x>0$

No. Date

<u>Περίπτωση:</u>

$y'(x)+py(x)=q \quad , p,q \in \mathbb{R} \quad , p\neq 0, q\neq 0.$

$e^{xp}y'(x)+e^{xp}py(x)=e^{px}\cdot q$

$\int (e^{xp}y(x))' dx = \int e^{px} q dx + C.$

$e^{xp}y(x) = e^{px}\frac{q}{p} + C \Rightarrow \boxed{y(x)=\frac{q}{p}+C\cdot e^{-xp}, x\in I}$

Αν $q=0 \rightsquigarrow y(x)=C\cdot e^{-px}, x\in I$

<u>Παράδειγμα:</u>

$y'+2y=\begin{cases} 1-|x|, & |x|\leq 1 \\ 0, & |x|>1 \end{cases} \quad , y(0)=5.$

$y'(x)+2y(x)=\begin{cases} 1-|x|, & -1\leq x\leq 1 \\ 0, & x>1, x<-1 \end{cases}$

Για $x>1, x\leq -1$

$y'(x)+2y(x)=0 \quad , |x|>1$

$e^{2x}y'(x)+e^{2x}\cdot 2y(x)=0 \Rightarrow \int (e^{2x}y(x))' dx$

$=\int \cancel{0}\, dx + C$

$e^{2x}y(x)=C \Rightarrow y(x)=C\cdot e^{-2x}$

$\rightsquigarrow y(0)=C\cdot e^{0} \Rightarrow \boxed{C=5}$

$y(x)=5\cdot e^{-2x}, |x|\geq 1$

$\boxed{-1 \leq x \leq 1}$

- $y'(x) + 2y(x) = 1 - x \quad , 0 < x \leq 1$
- $y'(x) + 2y(x) = 1 - x \quad , -1 \leq x \leq 0$

$\boxed{0 < x \leq 1}$

$$e^{2x} y'(x) + 2 \cdot e^{2x} y(x) = (1-x) e^{2x} = e^{2x} - x \cdot e^{2x}$$

$$\int (e^{2x} y(x))' dx = \int (1-x) e^{2x} dx + C$$

$$\int (1-x) e^{2x} dx = \left[\frac{e^{2x}}{2}(1-x)\right] + \int \frac{e^{2x}}{2} dx$$

$$= \frac{e^{2x}}{2}(1-x) + \frac{e^{2x}}{4} + C$$

Άρα

$$e^{2x} y(x) = \frac{e^{2x}}{2}(1-x) + \frac{e^{2x}}{4} + C$$

$$\Rightarrow y(x) = \frac{(1-x)}{2} + \frac{1}{4} + C \cdot e^{x}$$

$$x = 0 \rightsquigarrow 5 = \frac{1}{2} + \frac{1}{4} + C \Rightarrow C = \frac{20-2-1}{4} \Rightarrow \boxed{C = \frac{17}{4}}$$

$$y(x) = \frac{(1-x)}{2} + 17 \cdot e^{-x} + \frac{1}{4} \;, 0 < x \leq 1$$

- ομοίως $-1 \leq x \leq 0$ (άσκηση)

y συνεχής στο $x_0 = 1$

- $\lim_{x \to 1^+} y(x) = \lim_{x \to 1^+} 0 = 0$
- $\lim_{x \to 1^-} y(x) = \lim_{x \to 1^-} 0 = 0$
- $y(1) = 1 - |1| = 1 - 1 = 0$

y συνεχής στο 1
ή με τον τύπο
$y(x) = e^{-2x}\left(y(0) + \int_0^x q(s) \cdot e^{2s} ds\right)$
$x \in \mathbb{R}$